



Subconvexity bounds for Rankin–Selberg L -functions for congruence subgroups

Yuk-Kam Lau ^a, Jianya Liu ^{b,1}, Yangbo Ye ^{c,*,2}

^a *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

^b *Department of Mathematics, Shandong University, 27 Shanda Nanlu, Jinan, Shandong 250100, China*

^c *Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA*

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Abstract

Estimation of shifted sums of Fourier coefficients of cusp forms plays crucial roles in analytic number theory. Its known region of holomorphy and bounds, however, depend on bounds toward the general Ramanujan conjecture. In this article, we extended such a shifted sum meromorphically to a larger half plane $\text{Re } s > 1/2$ and proved a better bound. As an application, we then proved a subconvexity bound for Rankin–Selberg L -functions which does not rely on bounds toward the Ramanujan conjecture: Let f be either a holomorphic cusp form of weight k , or a Maass cusp form with Laplace eigenvalue $1/4 + k^2$, for $\Gamma_0(N)$. Let g be a fixed holomorphic or Maass cusp form. What we obtained is the following bound for the L -function $L(s, f \otimes g)$ in the k aspect:

$$L(1/2 + it, f \otimes g) \ll k^{1-1/(8+4\theta)+\varepsilon},$$

where θ is from bounds toward the generalized Ramanujan conjecture. Note that a trivial $\theta = 1/2$ still yields a subconvexity bound.

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* Corresponding author.

E-mail addresses: yklau@maths.hku.hk (Y.-K. Lau), jyliu@sdu.edu.cn (J. Liu), yey@math.uiowa.edu (Y. Ye).

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1. Introduction

Let f be a holomorphic Hecke eigenform for $\Gamma_0(\mathcal{N})$ of weight k , and g a fixed holomorphic or Maass cusp form. Sarnak [12] proved that

$$L(1/2 + it, f \otimes g) \ll_{\mathcal{N}, t, g, \varepsilon} k^{576/601 + \varepsilon}, \quad (1.1)$$

while the convexity bound from Phragmén–Lindelöf principle is merely $\ll k^{1+\varepsilon}$. The proof of this subconvexity bound made use of a bound toward the Ramanujan conjecture with $\theta = 7/64$ (Kim and Sarnak [7]):

$$\begin{aligned} |\alpha_\pi^{(j)}(p)| &\leq p^\theta \quad \text{for } p \text{ at which } \pi \text{ is unramified,} \\ |\Re \mu_\pi^{(j)}(\infty)| &\leq \theta \quad \text{if } \pi \text{ is unramified at } \infty, \end{aligned} \quad (1.2)$$

where π is an automorphic cuspidal representation of $GL_2(\mathbb{Q}_\mathbb{A})$ with unitary central character and local Hecke eigenvalues $\alpha_\pi^{(j)}(p)$ for $p < \infty$ and $\mu_\pi^{(j)}(\infty)$ for $p = \infty$, $j = 1, 2$. In terms of (1.2), the exponent in Sarnak's bound (1.1) can be written as $18/(19 - 2\theta) + \varepsilon$.

If f is a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4 + k^2$, Liu and Ye [9] proved similar subconvexity bounds. While the exponent $(3 + 2\theta)/4 + \varepsilon$ as claimed there does not hold because of a gap in Sections 4.14 and 4.15, the paper did prove a subconvexity bound

$$L(1/2 + it, f \otimes g) \ll_{\mathcal{N}, t, g, \varepsilon} k^{(15+2\theta)/16 + \varepsilon}, \quad (1.3)$$

as pointed out in the first sentence in Section 4.14. The proof of (1.3) is reproduced in Liu and Ye [10]. With $\theta = 7/64$, this means that we have $\ll k^{487/512 + \varepsilon}$.

Bounds (1.2) toward the Ramanujan conjecture played a crucial role in (1.1) and (1.3)—a non-trivial $\theta < 1/2$ is essential to get a subconvexity estimate. It is believed, however, that the Ramanujan conjecture is irrelevant to the Lindelöf hypothesis $L(1/2 + it, f \otimes g) \ll k^\varepsilon$; see Sarnak [13]. The main goal of the present paper is to give an evidence to this, i.e., to give a subconvexity bound which does not rely on bounds toward the Ramanujan conjecture.

Theorem 1. *Let f be a holomorphic Hecke eigenform for $\Gamma_0(\mathcal{N})$ of weight k , or a Maass Hecke eigenform for $\Gamma_0(\mathcal{N})$ with Laplace eigenvalue $1/4 + k^2$, and let g be a fixed holomorphic or Maass cusp form. Then*

$$L(1/2 + it, f \otimes g) \ll_{\mathcal{N}, t, g, \varepsilon} k^{1 - 1/(8+4\theta) + \varepsilon}. \quad (1.4)$$

Note that by taking the trivial $\theta = 1/2$, (1.4) yields a subconvexity bound $k^{9/10 + \varepsilon}$ which is already an improvement to (1.1) and (1.3).

Our Theorem 1 depends on new bounds for shifted convolution sums for Fourier coefficients of cusp forms; see Theorems 2 and 3 below. Let $g \in S_l(\Gamma_0(\mathcal{N}))$ be a holomorphic cusp form with Fourier expansion

$$g(z) = \sum_{n \geq 1} n^{(l-1)/2} \lambda_g(n) e(nz);$$

or let g be a Maass cusp form with Laplace eigenvalue $1/4 + l^2$ and Fourier expansion

$$g(z) = \sqrt{y} \sum_{n \neq 0} \lambda_g(n) K_{il}(2\pi|n|y) e(nx).$$

Assume that v_1, v_2 , and h are positive integers, and $s = \sigma + it$. We are going to estimate the shifted convolution sums

$$D_g(s, v_1, v_2, h) = \sum_{\substack{m, n > 0 \\ v_1 m - v_2 n = h}} \lambda_g(n) \bar{\lambda}_g(m) \left(\frac{\sqrt{v_1 v_2 m n}}{v_1 m + v_2 n} \right)^{l-1} (v_1 m + v_2 n)^{-s} \quad (1.5)$$

when g is a holomorphic cusp form, and

$$D_g(s, v_1, v_2, h) = \sum_{\substack{m, n \neq 0 \\ v_1 m - v_2 n = h}} \lambda_g(n) \bar{\lambda}_g(m) \left(\frac{\sqrt{v_1 v_2 |m n|}}{v_1 |m| + v_2 |n|} \right)^{2il} (v_1 |m| + v_2 |n|)^{-s} \quad (1.6)$$

when g is a Maass form, for $\sigma > 1/2$. Sarnak [12] proved that $D_g(s, v_1, v_2, h)$ extends to a holomorphic function on $\sigma > 1/2 + \theta$, and has the upper bound estimates

$$D_g(s, v_1, v_2, h) \ll_{g, \varepsilon} (v_1 v_2)^{1/2+\varepsilon} |h|^{1/2+\theta-\sigma+\varepsilon} (1+|t|)^3 \quad (1.7)$$

if g is holomorphic, and

$$D_g(s, v_1, v_2, h) \ll_{g, \varepsilon} (v_1 v_2)^{1/2+\varepsilon} |h|^{1/2+\theta-\sigma+\varepsilon} (1+|t|)^3 + |h|^{1-\sigma} \quad (1.8)$$

if g is Maass. The last term in (1.8) was removed by Blomer [2].

To get subconvexity independent of bounds toward the generalized Ramanujan conjecture, we need to continue $D_g(s, v_1, v_2, h)$ further to $\sigma > 1/2$, which is achieved in our Theorems 2 and 3 below.

Theorem 2. *Let g be a holomorphic cusp form of weight l , and $D_g(s, v_1, v_2, h)$ defined by (1.5). Then the function $D_g(s, v_1, v_2, h)$ admits an analytic continuation to a meromorphic function on $\sigma > 1/2$, with at most a finite number of poles $s_j \in (1/2, 1/2 + \theta]$ due to possible exceptional eigenvalues $\lambda_j = s_j(1 - s_j)$ of the Laplacian Δ . Moreover, for $\sigma \geq 1/2 + \varepsilon$ and $|t| \geq 1$, we have*

$$D_g(s, v_1, v_2, h) \ll_{v_1, v_2, g, \varepsilon} |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{1+\varepsilon}. \quad (1.9)$$

Theorem 3. *Let g be a Maass cusp form with eigenvalue $1/4 + l^2$, and $D_g(s, v_1, v_2, h)$ defined by (1.6). Then the function $D_g(s, v_1, v_2, h)$ admits an analytic continuation to a meromorphic function on $\sigma > 1/2$, with at most a finite number of poles $s_j \in (1/2, 1/2 + \theta]$ due to possible exceptional eigenvalues $\lambda_j = s_j(1 - s_j)$ of the Laplacian Δ . Moreover, for $\sigma \geq 1/2 + \varepsilon$ and $|t| \geq 1$, we have*

$$D_g(s, v_1, v_2, h) \ll_{v_1, v_2, g, \varepsilon} |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{1+\varepsilon} + |h|^{1-\sigma}. \quad (1.10)$$

Remarks.

- (1) Theorems 2 and 3 state that, if we allow the occurrence of poles, then $D_g(s, v_1, v_2, h)$ can be meromorphically continued to the bigger region $\sigma > 1/2$. It is this analytic continuation that enables us to establish subconvexity without using bounds toward the generalized Ramanujan conjecture.
- (2) The possible poles, if appear, should be on the segment $[1/2, 1]$. Their contribution is not covered by (1.9) or (1.10), since Theorems 2 and 3 require $|t| \geq 1$. In practice, however, contribution from these possible poles is not needed; see the argument after (3.6), and between (5.18) and (5.19), for details.
- (3) In (1.9) and (1.10), we also save in the t -aspect by reducing the $(1 + |t|)^3$ in (1.7) and (1.8) to $|t|^{1+\varepsilon}$. This saving comes from application of the mean-square theorems of Good [5] and Bernstein and Reznikov [1], instead of Sarnak's term-wise bound [11].
- (4) In (1.10), the term $|h|^{1-\sigma}$ will not affect our subconvexity bounds, as its contribution is small when we work in the larger half-plane $\sigma > 1/2$. Recall that when we restricted ourselves to $\sigma > 1/2 + \theta$ as in [9,12], this term $|h|^{1-\sigma}$ in (1.8) did weaken the final subconvexity bounds.

An early version of this paper was finished in October, 2004, and posted on the web page of the last author (<http://www.math.uiowa.edu/~yey/number.html>). On March 27, 2005, Blomer kindly sent us his manuscript [3], in which he got a better exponent $(6 - 2\theta)/(7 - 4\theta)$. It seems, however, that our meromorphic continuation and estimation of $D_g(s, v_1, v_2, h)$ is new and of interest in its own right. Moreover, our subconvexity bounds in Theorem 1 are still interesting as they do not depend on bounds toward the generalized Ramanujan conjecture.

2. Spectral theory on $L^2(\Gamma \backslash \mathbb{H})$

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, and denote by $L^2(\Gamma \backslash \mathbb{H})$ the L^2 -space of automorphic functions of weight 0 with respect to the Petersson inner product

$$\langle f_1, f_2 \rangle = \int_{\Gamma \backslash \mathbb{H}} f_1(z) \overline{f_2(z)} \frac{dx dy}{y^2}.$$

Also, we denote the non-Euclidean Laplacian by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

By the Maass–Selberg theory (see Deshouillers and Iwaniec [4, p. 227]), $L^2(\Gamma \backslash \mathbb{H})$ admits a spectral decomposition with respect to Δ . The spectrum of Δ consists of two components: The discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, and the continuous spectrum covering the segment $[1/4, \infty)$. Each eigenvalue in the discrete spectrum has finite order, and $\lambda_j \rightarrow \infty$ as $j \rightarrow \infty$. Moreover, there are two types of eigenvalues: $0 < \lambda_j < 1/4$ which are called exceptional, and $\lambda_j \geq 1/4$. The famous Selberg conjecture asserts that there is no exceptional eigenvalue for congruence groups, but the currently best known result is $\lambda_1 \geq 1/4 - \theta^2$, where θ is the value in (1.2), due to Kim and Sarnak [7]. Write $\lambda_j = s_j(1 - s_j)$ and $s_j = 1/2 + it_j$ where

$$0 < it_j \leq \theta \quad \text{if } \lambda_j \text{ is exceptional, and } \quad t_j \in [0, \infty) \quad \text{otherwise.} \quad (2.1)$$

According to Weyl's law, the number of eigenvalues up to T is

$$\#\{j: t_j \leq T\} = cT^2 + O(T \log T) \quad (2.2)$$

for some constant $c > 0$.

Let $\{\phi_0, \phi_1, \dots\}$ be an orthonormal basis of the eigenfunctions for the discrete spectrum. For a cusp \mathfrak{a} , denote by $\{E_{\mathfrak{a}}(z, 1/2 + i\tau): \tau \in \mathbb{R}\}$ the corresponding Eisenstein series which composes the eigenpacket for the continuous part. Then for any $f \in L^2(\Gamma \setminus \mathbb{H})$, we have a spectral expansion

$$f(z) = \sum_{j \geq 0} \langle f, \phi_j \rangle \phi_j(z) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}}(\cdot, 1/2 + i\tau) \rangle E_{\mathfrak{a}}(z, 1/2 + i\tau) d\tau, \quad (2.3)$$

where the summation $\sum_{\mathfrak{a}}$ runs over all cusps of Γ . Note that there are $O_{\Gamma}(1)$ cusps. While (2.3) only converges in the L^2 -sense, we may use it to compute an inner product with a function $V(z)$ to be defined later in Section 3.

Both $\phi_j(z)$ and $E_{\mathfrak{a}}(z, s)$ have Fourier series expansions: For $z = x + iy$,

$$\phi_j(z) = \sqrt{y} \sum_{m \neq 0} \rho_j(m) K_{it_j}(2\pi|m|y) e(mx),$$

and

$$\begin{aligned} E_{\mathfrak{a}}(z, s) &= \delta_{\mathfrak{a}\infty} y^s + \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \rho_{\mathfrak{a}}(s, 0) y^{1-s} \\ &\quad + \frac{2\pi^s \sqrt{y}}{\Gamma(s)} \sum_{m \neq 0} |m|^{s-1/2} \rho_{\mathfrak{a}}(s, m) K_{s-1/2}(2\pi|m|y) e(mx), \end{aligned}$$

where $\delta_{\mathfrak{a}\infty} = 1$ if $\mathfrak{a} = \infty$ and 0 otherwise. As in Sarnak [12, (A.16)], we can choose $\{\phi_j\}$ to be the Hecke eigenforms such that

$$\rho_j(m) \ll_{\varepsilon} \frac{(m\mathcal{N}t_j)^{\varepsilon}}{\sqrt{\mathcal{N}}} \cosh\left(\frac{\pi t_j}{2}\right) m^{\theta}. \quad (2.4)$$

3. Proof of Theorem 2

This section is devoted to $D_g(s, v_1, v_2, h)$ defined in (1.5) via spectral theory by modifying the argument in Sarnak [12, Appendix].

Let g be a holomorphic cusp form on $\Gamma_0(\mathcal{N})$ of weight l . Write $\Gamma = \Gamma_0(\mathcal{N}v_1v_2)$ and

$$V(z) = y^l g(v_1 z) \overline{g(v_2 z)}.$$

Then V is a Γ -invariant function rapidly decreasing at the cusps of Γ , and $V \in L^2(\Gamma \setminus \mathbb{H})$. Define the Poincaré series $U_h(z, s)$ for Γ by

$$U_h(z, s) = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \Im(\gamma z)^s e(-h\Re(\gamma z)),$$

where h is a positive integer, and $e(x) = e^{2\pi i x}$. By the standard unfolding method, $D_g(s, v_1, v_2, h)$ can be expressed in terms of the inner product (see [12, p. 444])

$$D_g(s, v_1, v_2, h) = (2\pi)^{s+l-1} (v_1 v_2)^{(l-1)/2} \frac{\langle U_h(\cdot, s), V \rangle}{\Gamma(s+l-1)}. \quad (3.1)$$

Since V is square-integrable (though not U_h) and $\Gamma \setminus \mathbb{H}$ is of finite volume, Parseval's identity applies, and therefore

$$\begin{aligned} \langle U_h(\cdot, s), V \rangle &= \sum_{j \geq 1} \langle U_h(\cdot, s), \phi_j \rangle \overline{\langle V, \phi_j \rangle} \\ &\quad + \frac{1}{4\pi} \sum_{\alpha} \int_{-\infty}^{\infty} \langle U_h(\cdot, s), E_{\alpha}(\cdot, 1/2 + i\tau) \rangle \overline{\langle V, E_{\alpha}(\cdot, 1/2 + i\tau) \rangle} d\tau. \end{aligned} \quad (3.2)$$

Note that $\langle U_h, \phi_0 \rangle = 0$. In view of (3.1), one may investigate the right side of (3.2) for the properties of $D_g(s, v_1, v_2, h)$. To this end, we need to evaluate some inner products.

Lemma 3.1. *We have*

$$\langle U_h(\cdot, s), \phi_j \rangle = \frac{\pi^{1/2-s} \overline{\rho_j(-h)}}{4|h|^{s-1/2}} \Gamma\left(\frac{s-1/2+it_j}{2}\right) \Gamma\left(\frac{s-1/2-it_j}{2}\right),$$

and

$$\begin{aligned} \langle U_h(\cdot, s), E_{\alpha}(\cdot, 1/2 + i\tau) \rangle &= \frac{\pi^{1-s-i\tau}}{\Gamma(1/2 - i\tau)} \frac{\overline{\rho_{\alpha}(1/2 + i\tau, -h)}}{2|h|^{s-1/2+i\tau}} \\ &\quad \times \Gamma\left(\frac{s-1/2+i\tau}{2}\right) \Gamma\left(\frac{s-1/2-i\tau}{2}\right). \end{aligned}$$

This follows readily from [12, (A12)] and the formula

$$\int_0^{\infty} K_{\nu}(t) t^{\mu-1} dt = 2^{\mu-2} \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right)$$

for $\Re \mu > |\Re \nu|$, by Watson [14, p. 388(8)].

Lemma 3.1 with (2.1) yields immediately that each summand on the right side of (3.2) is holomorphic in $\sigma > 1/2 + \theta$. Using the estimate of individual $\langle V, \phi_j \rangle$ developed in [11], Sarnak [12, Theorem A.1] concluded that $D_g(s, v_1, v_2, h)$ extends to a holomorphic function on $\sigma > 1/2 + \theta$ and has the upper estimate (1.7). But, indeed, if we allow the occurrence of poles, $D_g(s, v_1, v_2, h)$ can be meromorphically continued to a bigger region. Furthermore, we may refine Sarnak's estimate in the t -aspect via the mean square estimate in Good [5] rather than the term-wise bound.

In view of (3.1), (3.2), and Lemma 3.1, we introduce the following functions:

$$B_j(h, s) = (2\pi)^{s+l-1} (v_1 v_2)^{(l-1)/2} \frac{\langle U_h(\cdot, s), \phi_j \rangle}{\Gamma(s+l-1)},$$

$$C_a(h, s, \tau) = (2\pi)^{s+l-1} (v_1 v_2)^{(l-1)/2} \frac{\langle U_h(\cdot, s), E_a(\cdot, 1/2 + i\tau) \rangle}{\Gamma(s+l-1)},$$

and denote by $R_h(s)$ the sum over the exceptional eigenvalues,

$$R_h(s) = \frac{(v_1 v_2)^{(l-1)/2} 2^{s+l-3} \pi^{l-1/2}}{\Gamma(s+l-1)} \sum_{1/2 < s_j \leq 1/2 + \theta} \frac{\overline{\rho_j(-h)}}{|h|^{s-1/2}} \\ \times \Gamma\left(\frac{s-s_j}{2}\right) \Gamma\left(\frac{s-(1-s_j)}{2}\right) \overline{\langle V, \phi_j \rangle}. \quad (3.3)$$

Then for $\sigma > 1$, we have

$$D_g(s, v_1, v_2, h) - R_h(s) \\ = \sum_{j: t_j \geq 0} B_j(h, s) \overline{\langle V, \phi_j \rangle} + \frac{1}{4\pi} \sum_a \int_{-\infty}^{\infty} C_a(h, s, \tau) \overline{\langle V, E_a(\cdot, 1/2 + i\tau) \rangle} d\tau. \quad (3.4)$$

As $R_h(s)$ is a finite sum and

$$\langle V, \phi_j \rangle \ll \|V\| \|\phi_j\| \ll_{v_1, v_2, g} 1,$$

$R_h(s)$ is analytic in the half-plane $\sigma > 0$ except for poles at s_j and $1 - s_j$. Inserting (2.4) into (3.3), and then applying

$$|\Gamma(\sigma + it)| \asymp |t|^{\sigma-1/2} e^{-\pi|t|/2} \quad (3.5)$$

which holds for $-2 \leq \sigma \leq 2$ and $|t| \geq 1$, we deduce, for $1/2 \leq \sigma \leq 2$ and $|t| \geq 1$,

$$R_h(s) \ll_{v_1, v_2, g} |h|^{1/2+\theta-\sigma+\varepsilon}. \quad (3.6)$$

However, the above estimate is not true in the region $1/2 \leq \sigma \leq 2$ and $|t| \leq 1$, since the factor $\Gamma((s-s_j)/2)$ in (3.3) has a pole at $s = s_j = 1/2 + it_j$ with $0 < it_j \leq \theta$ as in (2.1). Obviously, these poles lie in the interval $[1/2, 1/2 + \theta] \subset [1/2, 1]$. This is why we require $|t| \geq 1$ in Theorems 2 and 3.

By Lemma 3.1, $B_j(h, s)$ (when $t_j \geq 0$) and $C_a(h, s, \tau)$ are holomorphic in $\sigma > 1/2$. The right side of (3.4) is analytically continued to a holomorphic function on $\sigma > 1/2$, provided that uniform convergence on compact sets is justified.

From (2.4) and (3.5), we infer that for $1/2 + \varepsilon \leq \sigma \leq 3/2$,

$$e^{-\pi t_j/2} B_j(h, s) \ll_{l, N} |h|^{1/2-\sigma+\theta+\varepsilon} t_j^\varepsilon \frac{(1+||t|-t_j|)^{\sigma/2-3/4}}{(1+|t|)^{\sigma/2+l-3/4}} e^{-\pi(|t-t_j|+|t+t_j|-2|t|)/4} \quad (3.7)$$

and, with Blomer [2, Lemma 3.4] in place of (2.4),

$$e^{-\pi|\tau|/2} C_a(h, s, \tau) \ll_{l, \mathcal{N}} |h|^{1/2-\sigma+\varepsilon} (1+|\tau|)^{\varepsilon} \frac{(1+||t|-\tau|)^{\sigma/2-3/4}}{(1+|t|)^{\sigma/2+l-3/4}} e^{-\pi(|t-\tau|+|t+\tau|-2|t|)/4}. \quad (3.8)$$

To verify the uniform convergence of (3.4) on compact sets, we assume $l \geq 4$ and invoke Good [5, Theorem 1]. The function V is different from the form of f there; nonetheless, Good's result still covers our case. This is because his proof applies to $f_l(z) = y^k F(z) P_l(z)$ where F and P_l are a cusp form and a Poincaré series for Γ , respectively; see [5, (3.2)] and [5, §4]. Note that $g(\nu_1 z)$ and $g(\nu_2 z)$ are cusp forms for Γ , and therefore $g(\nu_2 z)$ can be written as a linear combination of the Poincaré series. Hence,

$$\sum_{t_j \leq T} |\langle V, \phi_j \rangle|^2 e^{\pi t_j} + \frac{1}{4\pi} \sum_a \int_{-T}^T |\langle V, E_a(\cdot, 1/2 + i\tau) \rangle|^2 e^{\pi|\tau|} d\tau \ll T^{2l}. \quad (3.9)$$

Estimate (3.9) is also valid for $l = 2$, by Krötz and Stanton [8]. Plainly $|t - \tau| + |t + \tau| - 2|t| \geq |\tau|$ if $|\tau| \geq 2|t|$.

Thus, by (3.7) and Weyl's law (2.2), for $T \geq 2|t|$,

$$\begin{aligned} \sum_{t_j \geq T} e^{-\pi t_j} |B_j(h, s)|^2 &\ll |h|^{1+2\theta-2\sigma+\varepsilon} (1+|t|)^{3/2-\sigma-2l} \sum_{t_j \geq T} e^{-\pi t_j/4} \\ &\ll |h|^{1+2\theta-2\sigma+\varepsilon} e^{-3T/4}. \end{aligned} \quad (3.10)$$

Also, by (3.8), we have, for $T \geq 2|t|$,

$$\int_{|\tau| \geq T} e^{-\pi|\tau|} |C_a(h, s, \tau)|^2 d\tau \ll |h|^{1-2\sigma+\varepsilon} e^{-3T/4} \ll |h|^{1+2\theta-2\sigma+\varepsilon} e^{-3T/4}. \quad (3.11)$$

Now assume $T_0 \geq 2|t|$. Dividing dyadically and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\sum_{j: T_0 < t_j \leq T} |B_j(h, s) \overline{\langle V, \phi_j \rangle}| \\ &\leq \sum_{0 \leq r \leq \log T} \sum_{j: Y_r < t_j \leq 2Y_r} |B_j(h, s) \overline{\langle V, \phi_j \rangle}| \\ &\leq \sum_{0 \leq r \leq \log T} \left(\sum_{j: Y_r < t_j \leq 2Y_r} e^{-\pi t_j} |B_j(h, s)|^2 \right)^{1/2} \left(\sum_{j: Y_r < t_j \leq 2Y_r} e^{\pi t_j} |\langle V, \phi_j \rangle|^2 \right)^{1/2}. \end{aligned} \quad (3.12)$$

Here Y_r denotes $2^r T_0$. The sums in the brackets of the last line of (3.12) are $\ll |h|^{1-2\sigma+2\theta+\varepsilon} \times e^{-3Y_r/4}$ and $\ll Y_r^{2l}$, respectively, by (3.10) and (3.9). Therefore (3.12) becomes

$$\begin{aligned} \sum_{j: T_0 < t_j \leq T} |B_j(h, s) \overline{\langle V, \phi_j \rangle}| &\ll |h|^{1/2+\theta-\sigma+\varepsilon} \sum_{0 \leq r \leq \log T} e^{-3Y_r/8} Y_r^l \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} e^{-T_0/4} \end{aligned} \quad (3.13)$$

for any $T \geq T_0$.

Similarly it follows from Cauchy–Schwarz’s inequality that

$$\begin{aligned} &\int_{T_0 \leq |\tau| \leq T} |C_a(h, s, \tau) \overline{\langle V, E_a(\cdot, 1/2 + i\tau) \rangle}| d\tau \\ &\ll \sum_{0 \leq r \leq \log T} \left(\int_{Y_r \leq |\tau| \leq 2Y_r} e^{-\pi|\tau|} |C_a(h, s, \tau)|^2 d\tau \right)^{1/2} \\ &\quad \times \left(\int_{Y_r \leq |\tau| \leq 2Y_r} e^{\pi|\tau|} |\langle V, E_a(\cdot, 1/2 + i\tau) \rangle|^2 d\tau \right)^{1/2} \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} e^{-T_0/4}. \end{aligned} \quad (3.14)$$

Consequently, for any fixed compact subset K in the half plane $\sigma > 1/2$, (3.13) and (3.14) hold uniformly for $s \in K$ when $T_0 \geq 2 \max_{s \in K} |t|$. By the Cauchy criterion, uniform convergence of (3.4) on compact sets is verified.

To complete the proof of Theorem 2, it remains to prove (1.9). Let us consider $\sigma \geq 1/2 + \varepsilon$ and $|t| \geq 1$. By (3.13) and (3.14) with $T_0 = 2|t|$, we see that the tail part in (3.4) yields

$$\begin{aligned} &\sum_{j: t_j \geq 2|t|} B_j(h, s) \overline{\langle V, \phi_j \rangle} + \frac{1}{4\pi} \sum_a \int_{|\tau| \geq 2|t|} C_a(h, s, \tau) \overline{\langle V, E_a(\cdot, 1/2 + i\tau) \rangle} d\tau \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} e^{-|t|/2}. \end{aligned} \quad (3.15)$$

As $|t - t_j| + |t + t_j| - 2|t|$ is always non-negative, we may apply (3.7) to $B_j(h, s)$, to get

$$\begin{aligned} &\sum_{j: t_j \leq 2|t|} |B_j(h, s) \overline{\langle V, \phi_j \rangle}| \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{3/4-\sigma/2-l+\varepsilon} \sum_{t_j \leq 2|t|} |\langle V, \phi_j \rangle| e^{\pi t_j/2} (1 + |t - t_j|)^{\sigma/2-3/4} \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{3/4-\sigma/2-l+\varepsilon} \left(\sum_{t_j \leq 2|t|} |\langle V, \phi_j \rangle|^2 e^{\pi t_j} \right)^{1/2} \left(\sum_{t_j \leq 2|t|} (1 + |t - t_j|)^{\sigma-3/2} \right)^{1/2} \\ &\ll |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{1+\varepsilon} \end{aligned}$$

by (3.9) and Weyl’s law (2.2). We remark that the precise form of (2.2) is needed to evaluate the last bracket.

Similarly, by (3.9),

$$\begin{aligned}
 & \sum_{\mathfrak{a}} \int_{-2|t|}^{2|t|} |C_{\mathfrak{a}}(h, s, \tau) \overline{\langle V, E_{\mathfrak{a}}(1/2 + i\tau) \rangle}| d\tau \\
 & \ll |h|^{1/2-\sigma+\varepsilon} |t|^{3/4-\sigma/2-l+\varepsilon} \sum_{\mathfrak{a}} \int_{-2|t|}^{2|t|} |\langle V, E_{\mathfrak{a}}(1/2 + i\tau) \rangle| e^{\pi|\tau|/2} (1 + |t - |\tau||)^{\sigma/2-3/4} d\tau \\
 & \ll |h|^{1/2-\sigma+\varepsilon} |t|^{3/4-\sigma/2-l+\varepsilon} \sum_{\mathfrak{a}} \left(\int_{-2|t|}^{2|t|} |\langle V, E_{\mathfrak{a}}(1/2 + i\tau) \rangle|^2 e^{\pi|\tau|} d\tau \right)^{1/2} \\
 & \quad \times \left(\int_{-2|t|}^{2|t|} (1 + |t - |\tau||)^{\sigma-3/2} d\tau \right)^{1/2} \\
 & \ll |h|^{1/2-\sigma+\varepsilon} |t|^{1/2+\varepsilon}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 & \sum_{j: t_j \leq 2|t|} B_j(h, s) \overline{\langle V, \phi_j \rangle} + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{|\tau| \leq 2|t|} C_{\mathfrak{a}}(h, s, \tau) \overline{\langle V, E_{\mathfrak{a}}(\cdot, 1/2 + i\tau) \rangle} d\tau \\
 & \ll |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{1+\varepsilon}.
 \end{aligned}$$

Inserting this, (3.15), and (3.6) into (3.4), we get (1.9), and hence Theorem 2.

4. Proof of Theorem 3

Let g be a Maass cusp form with eigenvalue $1/4 + l^2$, and the shifted sum $D_g(s, v_1, v_2, h)$ as in (1.6). Define

$$I = \langle U_h(\cdot, s), V \rangle$$

as in [12, (A31)], where $V(z) = g(v_1 z) \overline{g(v_2 \bar{z})}$. Then we follow the argument (3.2)–(3.8) in Section 3, and employ an inequality of the type in (3.9), which is available in Bernstein and Reznikov [1] or Krötz and Stanton [8]. Thus I has an analytic continuation to $\sigma \geq 1/2 + \varepsilon$, and I is bounded from above by the right side of (1.9) when $\sigma \geq 1/2 + \varepsilon$ and $|t| \geq 1$, i.e.

$$I \ll_{v_1, v_2, g, \varepsilon} |h|^{1/2+\theta-\sigma+\varepsilon} |t|^{1+\varepsilon}. \quad (4.1)$$

However, the integral I is not exactly the same as our desired $D_g(s, v_1, v_2, h)$.

To recover $D_g(s, v_1, v_2, h)$ from (4.1), we follow [12, (A34)–(A36)] to show that I is given by the product of some gamma factors and

$$\sum_{v_1 m - v_2 n = h} \frac{\lambda_g(m) \overline{\lambda_g(n)}}{(v_1 |m| + v_2 |n|)^s} \left(\frac{\sqrt{v_1 v_2 |mn|}}{v_1 |m| + v_2 |n|} \right)^{2il} F\left(\frac{s}{2} + il, \frac{1}{2} + il, \frac{s}{2} + \frac{1}{2}, \left(\frac{v_1 |m| - v_2 |n|}{v_1 |m| + v_2 |n|} \right)^2\right)$$

for $\sigma > 1$. Then we proceed with the steps in [12, (A37)], i.e. apply the Taylor expansion to the hypergeometric function F . Note that the O -term produced from the tail of the hypergeometric function is absolutely convergent for $\sigma > 0$ and bounded by $O(|h|^{1-\sigma})$. This can be seen from the estimate (see Iwaniec [6, Theorem 3.2])

$$\sum_{m \ll X} |\lambda_g(m)|^2 \ll X.$$

From this and (4.1), we deduce that, for Maass form g , $D_g(s, \nu_1, \nu_2, h)$ is meromorphic on the half-plane $\sigma > 1/2$ with poles arising from the exceptional eigenvalues, and, for $\sigma \geq 1/2 + \varepsilon$ and $|t| \geq 1$,

$$D_g(\sigma + it, \nu_1, \nu_2, h) \ll_{\nu_1, \nu_2, g, \varepsilon} |h|^{1/2 + \theta - \sigma + \varepsilon} |t|^{1 + \varepsilon} + |h|^{1 - \sigma}.$$

This proves Theorem 3.

5. Proof of Theorem 1

We will give a proof of Theorem 1 for f being Maass in this and the next sections. The holomorphic case can be treated likewise.

Following [9, 12] closely, we take $K^{1/2 + \Delta} \leq L \leq K^{1 - \Delta}$ and $K^{2 - \Delta} \leq Y \leq K^{2 + \varepsilon}$ for a small $\Delta > 0$. Let $\{f_j\}$ be an orthonormal basis, consisting of Hecke eigenforms, of the space of Maass cusp forms. Denote by $1/4 + k_j^2$ the Laplace eigenvalue for f_j . Let H be a smooth function of compact support in the interval $(1, 2)$. In virtue of [9, (5.1)], the proof of Theorem 1 is reduced to verifying

$$\sum_{K - L \leq k_j \leq K + L} |S_Y(f_j)|^2 \ll LKY^{1 + \varepsilon}, \quad (5.1)$$

where

$$S_Y(f) = \sum_n \lambda_f(n) \lambda_g(n) H\left(\frac{n}{Y}\right).$$

We follow [9] up to [9, (4.10)]; it then remains to bound

$$\begin{aligned} \tilde{T}_{\mu, \nu, j}^{(\eta)}(Y) &= \frac{Y^{(\mu - j + 1)/2}}{L^{2\nu - 1}} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{\eta_3 L^2}{2\pi i \sqrt{Y}} \right)^k \sum_{\substack{c \leq Y/(LK^{1 - \varepsilon}) \\ \mathcal{N}|c}} c^{j + k - \mu - 1} \\ &\quad \times \sum_{n, r \geq 1} \frac{\lambda_g(n) \bar{\lambda}_g(r)}{n^{(k - \mu)/2 + 1/4} r^{j/2 + 1/4}} H\left(\frac{n}{Y}\right) \\ &\quad \times B_{\eta, Y, c}^{(\mu, \nu, j, k)}(n, r) \sum_{\substack{z \bmod c \\ (z, c) = 1}} e\left(\frac{\bar{z}(n - r)}{c}\right), \end{aligned} \quad (5.2)$$

where

$$\begin{aligned}
B_{\eta,Y,c}^{(\mu,v,j,k)}(n,r) &= \int_0^\infty e\left(\frac{2\sqrt{wY}(\eta_1\sqrt{n} + \eta_2\sqrt{r})}{c} - \frac{\eta_3 K^2 c}{4\pi^2\sqrt{wYn}}\right) \\
&\quad \times \left\{ L\left(u^{2k} \frac{d^{2v}}{du^{2v}}(uh(u))\right)^\wedge \left(\frac{\eta_3 L K c}{2\pi^2\sqrt{wYn}}\right) \right. \\
&\quad \left. + K(u^{2k} h^{(2v)}(u))^\wedge \left(\frac{\eta_3 L K c}{2\pi^2\sqrt{wYn}}\right) \right\} \frac{\bar{H}(w)}{w^{(j+k-\mu+1)/2}} dw. \quad (5.3)
\end{aligned}$$

Here $0 \leq 2\mu \leq v < N$, $0 \leq j < 2N$, with N a suitably large constant at our disposal; $\eta = (\eta_1, \eta_2, \eta_3)$ with $\eta_j = \pm 1$. The sum over c runs through multiples of \mathcal{N} as we are considering the congruence subgroup case.

In (5.2), we change variables $h = r - n$, and apply the well-known formula (see, e.g., [6, (2.26)]) for the Ramanujan sum

$$\sum_{\substack{z \bmod c \\ (z,c)=1}} e\left(\frac{zn}{c}\right) = \sum_{\delta|(c,n)} \mu\left(\frac{c}{\delta}\right) \delta.$$

Then we obtain, instead of [9, (4.11)],

$$\begin{aligned}
\tilde{T}_{\mu,v,j}^{(\eta)}(Y) &\ll \frac{Y^{(\mu-j+1)/2}}{L^{2v-1}} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{L^2}{2\pi\sqrt{Y}}\right)^k \sum_{\delta \leq Y/(LK^{1-\varepsilon})} \delta \\
&\quad \times \sum_{\substack{c \leq Y/(LK^{1-\varepsilon}) \\ \delta|c}} c^{j+k-\mu-1} \sum_{\substack{|h| \leq Y \\ \delta|h}} |P(c, h, Y)|, \quad (5.4)
\end{aligned}$$

where we have relaxed the condition $\mathcal{N}|c$ and denoted

$$P(c, h, Y) = \sum_{n > \max(0, -h)} \frac{\lambda_g(n) \bar{\lambda}_g(n+h)}{n^{(k-\mu)/2+1/4} (n+h)^{j/2+1/4}} H\left(\frac{n}{Y}\right) B_{\eta,Y,c}^{(\mu,v,j,k)}(n, n+h) \quad (5.5)$$

for $0 \leq 2\mu \leq v < N$ and $0 \leq j < 2N$.

Unlike in [9, §4.7–4.8] where the signs of η_1 and η_2 are considered separately, here we give a uniform treatment. In fact, the argument of [9, §4.8] holds for all possible signs of η_1 and η_2 , and therefore, according to [9, (4.13)–(4.15)], equality (5.5) above can be rewritten by Mellin transform as

$$P(c, h, Y) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} D_g(s, 1, 1, h) \tilde{G}(s) ds \quad (5.6)$$

for any $\sigma > 1$, where

$$\tilde{G}(s) = (2Y)^s Y^{(\mu-j-k-1)/2} \int_0^\infty G_0(z) \left(z + \frac{h}{2Y}\right)^{s-1} dz \quad (5.7)$$

with

$$G_0(z) = \left(\frac{2z + h/Y}{\sqrt{z(z + h/Y)}} \right)^{l-1} z^{(\mu-k-j-1)/2} \left(1 + \frac{h}{Yz} \right)^{-j/2-1/4} \\ \times H(z) B_{\eta, Y, c}^{(\mu, \nu, j, k)}(zY, zY + h). \quad (5.8)$$

Inserting (5.3) into (5.8), and then inserting (5.8) into (5.7), we have

$$\tilde{G}(s) = 2K(2Y)^s Y^{(\mu-j-k-1)/2} \int_0^\infty \int_0^\infty \left(\frac{2z + h/Y}{\sqrt{z(z + h/Y)}} \right)^{l-1} z^{(\mu-j-k-1)/2} \\ \times \left(1 + \frac{h}{Yz} \right)^{-j/4-1/4} e \left(\frac{2w\sqrt{Y}(\eta_1\sqrt{Yz} + \eta_2\sqrt{Yz+h})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 w Y \sqrt{z}} \right) \\ \times \left\{ \frac{L}{K} \left(u^{2k} \frac{d^{2\nu}}{du^{2\nu}} (uh(u)) \right)^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y \sqrt{z}} \right) + (u^{2k} h^{(2\nu)}(u))^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y \sqrt{z}} \right) \right\} \\ \times \frac{1}{w^{j+k-\mu}} \left(z + \frac{h}{2Y} \right)^{s-1} \bar{H}(w^2) H(z) dw dz,$$

after replacing w by w^2 . Changing variables $w \rightarrow w/\sqrt{z}$,

$$\tilde{G}(s) = 2K(2Y)^s Y^{(\mu-j-k-1)/2} \int_0^\infty \int_0^\infty \left(\frac{2 + h/(Yz)}{\sqrt{1 + h/(Yz)}} \right)^{l-1} \left(1 + \frac{h}{Yz} \right)^{-j/4-1/4} \\ \times e \left(\frac{2wY(\eta_1 + \eta_2\sqrt{1 + h/(Yz)})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 w Y} \right) \\ \times \left\{ \frac{L}{K} \left(u^{2k} \frac{d^{2\nu}}{du^{2\nu}} (uh(u)) \right)^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y} \right) + (u^{2k} h^{(2\nu)}(u))^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y} \right) \right\} \\ \times \frac{1}{z} \frac{1}{w^{j+k-\mu}} \left(z + \frac{h}{2Y} \right)^{s-1} \bar{H} \left(\frac{w^2}{z} \right) H(z) dw dz. \quad (5.9)$$

We note that after changing variables, z disappeared from the terms within $\{\dots\}$ above. Therefore,

$$\tilde{G}(s) = 2K(2Y)^\sigma Y^{(\mu-j-k-1)/2} \int_0^\infty \int_0^\infty \left(\frac{2 + h/(Yz)}{\sqrt{1 + h/(Yz)}} \right)^{l-1} \left(1 + \frac{h}{Yz} \right)^{-j/4-1/4} \\ \times \left\{ \frac{L}{K} \left(u^{2k} \frac{d^{2\nu}}{du^{2\nu}} (uh(u)) \right)^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y} \right) + (u^{2k} h^{(2\nu)}(u))^\wedge \left(\frac{\eta_3 L K c}{2\pi^2 w Y} \right) \right\} \\ \times \frac{1}{z} \frac{1}{w^{j+k-\mu}} \left(z + \frac{h}{2Y} \right)^{\sigma-1} \bar{H} \left(\frac{w^2}{z} \right) H(z) e(\Phi(z, w, t)) dw dz, \quad (5.10)$$

where

$$\Phi(z, w, t) = \frac{t}{2\pi} \log(2Yz + h) + \frac{2wY(\eta_1 + \eta_2\sqrt{1 + h/(Yz)})}{c} - \frac{\eta_3 K^2 c}{4\pi^2 wY}. \quad (5.11)$$

We need the following lemma to estimate $\tilde{G}(s)$.

Lemma 5.1. Suppose $s = \sigma + it$ with $1/2 \leq \sigma \leq 2$ and $t \in \mathbb{R}$. Assume $c \leq Y/(LK^{1-\varepsilon})$.

(1) We have

$$\tilde{G}(s) \ll KY^{(\mu-j-k-1)/2+\sigma}. \quad (5.12)$$

(2) If $K^{2+3\varepsilon}c^2/Y \leq |h| \leq Y$, then for any $M \geq 1$,

$$\tilde{G}(s) \ll_{M,\varepsilon} K^{-M}. \quad (5.13)$$

(3) If $|h| \leq cK^\varepsilon$ and $c \geq K^{3\varepsilon}$, then (5.13) is valid.

(4) Let $cK^\varepsilon \leq |h| \leq K^{2+3\varepsilon}c^2/Y$. If $|t| \leq 1$, then (5.13) is valid.

(5) Let $c \leq |h| \leq Y$. If $|t| > K^\varepsilon|h|/c$, we have

$$\tilde{G}(s) \ll_{M,\varepsilon} |t|^{-M}. \quad (5.14)$$

(6) Let $|h| \leq cK^\varepsilon$. If $|t| \geq K^{2\varepsilon}$, then (5.14) also holds.

(7) Let $cK^\varepsilon \leq |h| \leq K^{2+3\varepsilon}c^2/Y$. Then (5.14) is valid when $|t| > (K/L)^{1+5\varepsilon/\Delta}$.

Proof. (1) The estimate (5.12) is trivial. Here we make use of the compact support of H and the rapid decay of $h(u)$, see [9, §4.3]. Actually (5.12) holds for any c ; the restriction $c \leq Y/(LK^{1-\varepsilon})$ is unnecessary.

(2) By (5.11), the derivative of Φ with respect to w is

$$\Phi'_w = \frac{2Y}{c} \left(\eta_1 + \eta_2 \sqrt{1 + \frac{h}{Yz}} \right) + \frac{\eta_3 K^2 c}{4\pi^2 w^2 Y}. \quad (5.15)$$

Since $|h| \leq Y$ and $z \in (1, 2)$, we have

$$\frac{2Y}{c} \left| \eta_1 + \eta_2 \sqrt{1 + \frac{h}{Yz}} \right| \sim \frac{2Y}{c} \left| \eta_1 + \eta_2 + \eta_2 \frac{h}{Yz} \right| \gg \frac{|h|}{c}$$

for all possible values of $\eta_1 = \pm 1, \eta_2 = \pm 1$, and therefore,

$$|\Phi'_w| \gg \frac{|h|}{c} - \frac{K^2 c}{Y} \gg \frac{|h|}{c} \geq K^{2\varepsilon}, \quad (5.16)$$

where we have applied the conditions $|h| \geq K^{2+3\varepsilon}c^2/Y$ and $Y \leq K^{2+\varepsilon}$.

Now we apply integration by parts to the integral in (5.10) with respect to w many times by integrating $e(\Phi(z, w, t))\Phi'_w$ and differentiating the rest of the integrand divided by Φ'_w . The differentiation of a function like

$$\frac{1}{w^{j+k-\mu}}\bar{H}\left(\frac{w^2}{z}\right)$$

yields a constant bound. The differentiation of the sum of the two Fourier transforms in (5.10) produces a factor

$$-\frac{\eta_3 L K c}{2\pi^2 w^2 Y} \ll K^\varepsilon,$$

by the assumption for c . The factor $1/\Phi'_w$ yields $O(K^{-2\varepsilon})$. The derivative of $1/\Phi'_w$ gives $O(K^{-2\varepsilon})$, since by (5.15) and (5.16),

$$\left(\frac{1}{\Phi'_w}\right)' = -\frac{\Phi''_w}{(\Phi'_w)^2} = \frac{1}{(\Phi'_w)^2} \frac{\eta_3 K^2 c}{2\pi^2 w^3 Y} \ll \frac{K^2 c/Y}{(|h|/c - K^2 c/Y)^2} \ll \left(\frac{|h|}{c}\right)^{-1} \ll K^{-2\varepsilon}.$$

Hence, each integration by parts produces a saving $O(K^{-\varepsilon})$. Doing this repeatedly produces a negligible $O(K^{-M})$ for arbitrary $M > 0$, and (5.13) follows.

(3) In this case, (5.13) can be proved similarly. The differences are the following: by $Y \leq K^{2+\varepsilon}$,

$$|\Phi'_w| \gg \frac{K^2 c}{Y} - \frac{h}{c} \gg K^{2\varepsilon} - K^\varepsilon \gg K^{2\varepsilon}$$

and

$$\left(\frac{1}{\Phi'_w}\right)' = \frac{1}{(\Phi'_w)^2} \frac{\eta_3 K^2 c}{2\pi^2 w^3 Y} \ll \frac{K^2 c/Y}{(K^2 c/Y - |h|/c)^2} \ll K^{-2\varepsilon}.$$

(4) Now we take derivative with respect to z . We have

$$\Phi'_z = \frac{t}{2\pi} \frac{1}{z + h/(2Y)} - \frac{w\eta_2}{c\sqrt{1+h/(Yz)}} \frac{h}{z^2}, \quad (5.17)$$

and therefore, for $|t| \leq 1$,

$$|\Phi'_z| \gg \frac{|h|}{c} - |t| \gg \frac{|h|}{c}.$$

Since $|h|/c \geq K^\varepsilon$, successive integration by parts yields the upper bound $O(K^{-M})$ for the z -integral.

(5) We note that in the present situation $|t| \geq K^\varepsilon |h|/c$, (5.17) gives

$$|\Phi'_z| \gg |t| - |h|/c \gg |t|(1 - K^{-\varepsilon}) \gg |t|,$$

which is $\gg K^\varepsilon$ for $|h| \geq c$. Therefore, the same argument as in (2) shows that successive integration by parts yields the upper bound $O(|t|^{-M})$ for the z -integral.

(6) When $|h| \leq cK^\varepsilon$ and $|t| \geq K^{2\varepsilon}$, we have

$$|\Phi'_z| \gg |t| - |h|/c \gg |t| - K^\varepsilon \gg |t|,$$

which is $\gg K^{2\varepsilon}$. By repeated integration by parts as in (2), we conclude that

$$\tilde{G}(\sigma + it) \ll_{M,\varepsilon} |t|^{-M}.$$

(7) We note that in the present situation,

$$\frac{|h|}{c} \leq \frac{K^{2+3\varepsilon}c}{Y} \leq \frac{K^{1+4\varepsilon}}{L} \leq \left(\frac{K}{L}\right)^{1+4\varepsilon/\Delta}.$$

Hence for $|t| \geq (K/L)^{1+5\varepsilon/\Delta}$, (5.16) gives

$$|\Phi'_z| \gg |t| - h/c \gg |t| - (K/L)^{1+4\varepsilon/\Delta} \gg |t|.$$

Therefore, the same argument shows that successive integration by parts yields the upper bound $O(|t|^{-M})$ for the z -integral. This completes the proof of Lemma 5.1. \square

6. The main estimation

We first estimate integral (5.6) in several steps.

Step 1. $|h| \leq cK^\varepsilon$ and $|t| \geq K^{2\varepsilon}$. By Lemma 5.1(6), we have

$$\frac{1}{2\pi} \int_{|t| > K^{2\varepsilon}} D_g(\sigma + it, 1, 1, h) \tilde{G}(\sigma + it) dt \ll_{M,\varepsilon} \int_{|t| > K^{2\varepsilon}} |t|^{-M} dt \ll_{M,\varepsilon} K^{2\varepsilon(1-M)}, \quad (6.1)$$

which is negligible.

Step 2. $|h| \leq cK^\varepsilon$, $|t| < K^{2\varepsilon}$, and $c \leq K^{3\varepsilon}$. We take $\sigma = 1 + \varepsilon$, so that $D_g(s, 1, 1, h) \ll_\varepsilon 1$, and apply the trivial estimate of $\tilde{G}(s)$ in Lemma 5.1(1). It follows that

$$P(c, h, Y) \ll KY^{(\mu-j-k+1)/2+\varepsilon}. \quad (6.2)$$

Step 3. $|h| \leq cK^\varepsilon$, $|t| < K^{2\varepsilon}$, and $c > K^{3\varepsilon}$. In view of $D_g(s, 1, 1, h) \ll_\varepsilon 1$ for $\sigma = 1 + \varepsilon$ and Lemma 5.1(3), we have

$$\frac{1}{2\pi} \int_{|t| < K^{2\varepsilon}} D_g(\sigma + it, 1, 1, h) \tilde{G}(\sigma + it) dt \ll_{M,\varepsilon} K^{-M} \int_{|t| < K^{2\varepsilon}} dt \ll_{M,\varepsilon} K^{2\varepsilon-M}, \quad (6.3)$$

which is negligible.

Step 4. $cK^\varepsilon < |h| \leq K^{2+3\varepsilon}c^2/Y$. Specifying $\sigma = 3/2$ in (5.6), we can write (5.6) as

$$P(c, h, Y) = \frac{1}{2\pi} \left\{ \int_{1 \leq |t| \leq (K/L)^{1+5\epsilon/\Delta}} + \int_{|t| \leq 1} + \int_{|t| \geq (K/L)^{1+5\epsilon/\Delta}} \right\} \\ \times D_g\left(\frac{3}{2} + it, 1, 1, h\right) \tilde{G}\left(\frac{3}{2} + it\right) dt.$$

As $D_g(3/2 + it, 1, 1, h) \ll 1$, the second integral is, by Lemma 5.1(4),

$$\int_{|t| \leq 1} K^{-M} dt \ll_{M,\epsilon} K^{-M}.$$

The last integral can be estimated by Lemma 5.1(7) as

$$\ll \int_{|t| \geq (K/L)^{1+5\epsilon/\Delta}} |t|^{-M} dt \ll_{M,\epsilon} (K/L)^{(1+5\epsilon/\Delta)(1-M)} \ll_{M,\epsilon} K^{(1-M)(\Delta+5\epsilon)},$$

which is negligible, because $L \leq K^{1-\Delta}$ for some $\Delta > 0$. Therefore, (5.6) becomes

$$P(c, h, Y) = \frac{1}{2\pi} \int_{1 \leq |t| \leq (K/L)^{1+5\epsilon}} D_g\left(\frac{3}{2} + it, 1, 1, h\right) \tilde{G}\left(\frac{3}{2} + it\right) dt + O_{M,\epsilon}(K^{-M}).$$

Now we write Theorems 2 and 3 in the form

$$D_g(s, v_1, v_2, h) \ll |h|^{1/2+\theta-\sigma+\epsilon} |t|^{1+\epsilon} + \delta(g) |h|^{1-\sigma},$$

where $\delta(g) = 1$ or 0 according as g is Maass or not. We remark that the term with $\delta(g)$ will make no significant contribution. We move the line segments to $\sigma = 1/2 + \epsilon$ with the horizontal parts controlled by Lemma 5.1(4). Applying the trivial bound in Lemma 5.1(1), we obtain, as $|h| \ll K^{2+3\epsilon} c^2/Y \ll (K/L)^{2+6\epsilon/\Delta}$,

$$P(c, h, Y) \ll KY^{(\mu-j-k)/2+\epsilon} \left\{ |h|^{\theta+\epsilon} \int_{1 \leq |t| \leq (K/L)^{1+5\epsilon/\Delta}} |t|^{1+\epsilon} dt \right. \\ \left. + \delta(g) |h|^{1/2+\epsilon} \int_{1 \leq |t| \leq (K/L)^{1+5\epsilon/\Delta}} dt \right\} \\ \ll KY^{(\mu-j-k)/2+\epsilon} \left(\frac{K}{L} \right)^{2+12\epsilon/\Delta} |h|^{\theta+\epsilon}. \quad (6.4)$$

Note that the term with $\delta(g)$ in (6.4) is smaller than the other term, and hence the contribution from the extra term $|h|^{1-\sigma}$ in (1.10) is absorbed in that of the main term.

Step 5. $K^{2+3\epsilon} c^2/Y < |h| \leq Y$. In this case, $|h| > cK^\epsilon$. We divide the integral in (5.6) into two integrals at $|t| = K^\epsilon |h|/c$. For the integral on $|t| > K^\epsilon |h|/c$, we use Lemma 5.1(5) and see that it is negligible:

$$\begin{aligned} & \frac{1}{2\pi} \int_{|t| > K^\varepsilon |h|/c} D_g(\sigma + it, 1, 1, h) \tilde{G}(\sigma + it) dt \\ & \ll_{M,\varepsilon} \int_{|t| > K^\varepsilon |h|/c} |t|^{-M} dt \ll_{M,\varepsilon} \left(\frac{K^\varepsilon |h|}{c} \right)^{1-M} \ll_{M,\varepsilon} K^{2\varepsilon(1-M)}. \end{aligned} \quad (6.5)$$

For the second integral, we use Lemma 5.1(2) and get

$$\begin{aligned} & \frac{1}{2\pi} \int_{|t| \leq K^\varepsilon |h|/c} D_g(\sigma + it, 1, 1, h) \tilde{G}(\sigma + it) dt \\ & \ll_{M,\varepsilon} K^{-M} \int_{|t| \leq K^\varepsilon |h|/c} dt \ll_{M,\varepsilon} K^{-M} \frac{K^\varepsilon |h|}{c} \ll_{M,\varepsilon} Y K^{\varepsilon-M}, \end{aligned} \quad (6.6)$$

which is again negligible.

Now we turn back to (5.4). By estimates (6.5) and (6.6) for $P(c, h, Y)$, we can rewrite (5.4) using a smaller range for h :

$$\begin{aligned} \tilde{T}_{\mu,v,j}^{(\eta)}(Y) & \ll \frac{Y^{(\mu-j+1)/2}}{L^{2v-1}} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{L^2}{2\pi\sqrt{Y}} \right)^k \sum_{\delta \leq Y/(LK^{1-\varepsilon})} \delta \\ & \times \sum_{\substack{c \leq Y/(LK^{1-\varepsilon}) \\ \delta|c}} c^{j+k-\mu-1} \sum_{\substack{|h| \leq K^{2+3\varepsilon} c^2/Y \\ \delta|h}} |P(c, h, Y)|. \end{aligned} \quad (6.7)$$

Using (6.1) through (6.4), we can compute the inner most sums over c and h when $L \geq K^{1/2+\Delta}$ as

$$\begin{aligned} & \sum_{\substack{c \leq Y/(LK^{1-\varepsilon}) \\ \delta|c}} c^{j+k-\mu-1} \sum_{\substack{|h| \leq K^{2+3\varepsilon} c^2/Y \\ \delta|h}} |P(c, h, Y)| \\ & \ll \sum_{\substack{c \leq K^{3\varepsilon} \\ \delta|c}} c^{j+k-\mu-1} \sum_{\substack{|h| \leq cK^\varepsilon \\ \delta|h}} KY^{(\mu-j-k+1)/2+\varepsilon} \\ & + \sum_{\substack{c \leq Y/(LK^{1-\varepsilon}) \\ \delta|c}} c^{j+k-\mu-1} \sum_{\substack{cK^\varepsilon < |h| \leq K^{2+3\varepsilon} c^2/Y \\ \delta|h}} KY^{(\mu-j-k)/2+\varepsilon} \left(\frac{K}{L} \right)^{2+12\varepsilon/\Delta} |h|^{\theta+\varepsilon} \\ & \ll \frac{1}{\delta^2} K^{1+4\varepsilon+3\varepsilon(j+k-\mu)} Y^{(\mu-j-k+1)/2+\varepsilon} \\ & + \frac{K^3 Y^{1+\theta+3\varepsilon}}{\delta^2 L^{4+2\theta}} \left(\frac{\sqrt{Y}}{LK} \right)^{j+k-\mu} K^{\varepsilon(j+k-\mu)+15\varepsilon+12\varepsilon/\Delta}. \end{aligned} \quad (6.8)$$

Inserting (6.8) into (6.7), we get

$$\begin{aligned} \tilde{T}_{\mu, \nu, j}^{(\eta)}(Y) &\ll \frac{Y^{(\mu-j+1)/2+\varepsilon}}{L^{2\nu-1}} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{L^2}{2\pi\sqrt{Y}} \right)^k \\ &\quad \times \left(K^{1+4\varepsilon+3\varepsilon(j+k-\mu)} Y^{(\mu-j-k+1)/2+\varepsilon} \right. \\ &\quad \left. + \frac{K^3 Y^{1+\theta+3\varepsilon}}{L^{4+2\theta}} \left(\frac{\sqrt{Y}}{LK} \right)^{j+k-\mu} K^{\varepsilon(j+k-\mu)+15\varepsilon+12\varepsilon/\Delta} \right), \end{aligned}$$

where the sum over δ contributed $O(Y^\varepsilon)$. Therefore,

$$\begin{aligned} \tilde{T}_{\mu, \nu, j}^{(\eta)}(Y) &\ll L^{1-2\nu} K^{1+4\varepsilon+3\varepsilon(j-\mu)} Y^{1+\mu-j+2\varepsilon} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{L^2 K^{3\varepsilon}}{2\pi Y} \right)^k \\ &\quad + \frac{K^{3+\varepsilon(15+12/\Delta)} Y^{(\mu-j+3)/2+\theta+4\varepsilon}}{L^{3+2\nu+2\theta}} \left(\frac{\sqrt{Y}}{LK^{1-\varepsilon}} \right)^{j-\mu} \sum_{0 \leq k \leq N} \frac{1}{k!} \left(\frac{L}{2\pi K^{1-\varepsilon}} \right)^k. \end{aligned} \quad (6.9)$$

The series in (6.9) converge as $N \rightarrow \infty$ and are bounded by $O(1)$, because we assumed that $L \leq K^{1-\Delta}$ for some $\Delta > 0$. In fact, we may take $\Delta = 1/10$, and hence $\varepsilon(15 + 12/\Delta)$ can be made arbitrarily small. Since $0 \leq 2\mu \leq \nu < N$ and $0 \leq j < 2N$, we can fix $N = 10$ and make $\varepsilon(j - \mu)$ arbitrarily small. By $0 \leq 2\mu \leq \nu$, $K^{1/2+\Delta} \leq L \leq K^{1-\Delta}$, and $K^{2-\Delta} \leq Y \leq K^{2+\varepsilon}$, we also get $Y^{\mu/2}/L^{2\nu} \ll 1$ and $Y^\mu/L^{2\nu} \ll 1$. Consequently, (6.9) is reduced to

$$\tilde{T}_{\mu, \nu, j}^{(\eta)}(Y) \ll LKY^{1+\varepsilon} + \frac{K^{3+\varepsilon} Y^{3/2+\theta+\varepsilon}}{L^{3+2\theta}} \quad (6.10)$$

for any small $\varepsilon > 0$. Thus (6.10) becomes the desired

$$\tilde{T}_{\mu, \nu, j}^{(\eta)}(Y) \ll LKY^{1+\varepsilon},$$

if $L \geq K^{1-1/(4+2\theta)+\varepsilon}$. This proves (5.1) via (5.2), and Theorem 1 follows from the argument of [9, §5].

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